Question 3.1: Matrix diagonalization

Consider a 2×2 Hermitian matrix:

$$M = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix},$$

with a and b real.

1. Show that the eigenvalues are

$$\lambda_{1,2} = \frac{\mathrm{Tr}M \pm \sqrt{(\mathrm{Tr}M)^2 - 4\det M}}{2}.$$

2. Prove that

$$(\mathrm{Tr}M)^2 \geq 4\det M.$$

This is needed in order to ensure that the eigenvalues are real.

3. Assume that c is real. In this case, the matrix M can be diagonalized by an orthogonal matrix O:

$$O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

We call θ the mixing angle. Show that

$$\tan 2\theta = \frac{2c}{b-a}.$$

4. Consider a general matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In general, it can be diagonalized by a bi-unitary transformation. That is

$$M_{\mathrm{diag}} = V_L M V_R^\dagger.$$

In order to find V_L and V_R, first show that $MM^\dagger$ and $M^\dagger M$ are Hermitian and that they are diagonalized by V_L and V_R, respectively. Then use the above formalism to find the diagonalization angles in the case where M is real.

Question 3.2: Non-Abelian gauge bosons

In this question you are asked to prove that the gauge bosons belong to the adjoint representation. Consider a field ϕ that transforms as an M-dimensional representation, R:

$$\phi \to U\phi, \qquad U = e^{iT_a\theta_a},$$

where a runs from one to the dimension of the group (for $SU(N)$, $a = 1, 2, \ldots, N^2 - 1$) and U and T^a are $M \times M$ matrices. We take θ_a to be independent of x_μ. This may seem weird, as the whole reason to introduce the gauge fields is to let θ_a depend on x_μ. Yet, once the gauge fields are introduced, they must transform also under the global symmetry, so $\theta_a = \text{const}(x_\mu)$ is just a special case. The covariant derivative is

$$D_\mu = \partial_\mu + igG_\mu, \qquad G_\mu \equiv G_\mu^a T_a.$$

1. Show that the infinitesimal transformation is

$$U = 1 + iT_a\theta_a + O(\theta_a^2).$$

2. Write the infinitesimal transformation of ϕ_k explicitly with the group indices.

3. In order to promote a global symmetry to a local one, $D_\mu\phi$ must transform the same way as ϕ, that is,

$$D_\mu\phi \to UD_\mu\phi. \qquad \text{Eq.(1)}$$

   Show that Eq.(1) implies that

$$G_\mu \to UG_\mu U^\dagger - \frac{1}{g}T_a\partial_\mu\theta_a. \qquad \text{Eq.(2)}$$

4. Show that Eq. (2), together with the algebra of the group, imply that for an infinitesimal gauge transformation

$$G_\mu^a \to G_\mu^a + \theta^c f^{abc} G_\mu^b - \frac{1}{g}\partial_\mu\theta^a.$$

Question 3.3: A mirror world

Consider the following model:

(i) The symmetry is a local $U(1)_{\text{EM}} \times U(1)_D$. We denote the corresponding gauge bosons by A_μ and C_μ, and their field strengths by $F_{\mu\nu}$ and $C_{\mu\nu}$, respectively.

(ii) There are four Weyl fermion fields:

$$e_L(-1,0), \qquad e_R(-1,0), \qquad d_L(0,-1), \qquad d_R(0,-1), \tag{3.29}$$

where the first number in the parenthesis is the charge under $U(1)_{\text{EM}}$ and the second is the charge under $U(1)_D$.

(iii) There are no scalars.

1. Write down the covariant derivative D_μ for a generic field with charge (q_{EM}, q_D), and then write it specifically for the four fermion fields. Use a normalization such that the coupling constants of the two groups is the same, that is, $g_{\text{EM}} = g_D = e$.

2. Draw the Feynman diagrams for this theory, *i.e.* draw all vertices from the interaction terms in the Lagrangian and write down their corresponding rules. Be sure to label the fields and take care of particle flow.

3. Find $\mathcal{L}_\psi$ and state what are the masses of the fermions and how many DoF each has.

4. Consider the term $C_{\mu\nu}F^{\mu\nu}$. (A term of this form is called "kinetic mixing" term.) Argue that this term is gauge invariant, Lorentz invariant, and has mass dimension $d = 4$.

Despite the fact that the kinetic mixing term is allowed we do not write it since we use canonical normalization. (It is a similar argument for why we do not write a term of the form $\bar{e}\gamma_u D^\mu \mu$.) Thus, we can write $\mathcal{L} = \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{D}}$ and the two sectors are completely decoupled. In particular, the process $e\bar{e} \to d\bar{d}$ is not allowed as it connects the two sectors.

5. We now add a scalar field $S(-1,+1)$ to the theory. Write down explicitly the covariant derivative $D_\mu S$.

6. Write down the most general coupling of S to the fermions (up to dimension-four terms). Write down the hermitian conjugate terms explicitly.

7. Explain why the theory violates parity. What are the consequences of imposing parity?

The model that we considered in this question is representative of a class of models where, in addition to a sector with the particles and interactions known to us, one adds a sector that is either completely decoupled from the first sector, or coupled to it only via very heavy degrees of freedom. The (almost) decoupled sector is often called a “dark sector.” Thus, C_μ would be called a “dark photon”, and its interactions can be termed as “dark QED.” We return to this model in Question 3.4.

Question 3.4: More on the dark photon

We consider a model that is an extension of the one discussed in Question 3.3:

(i) The symmetry is a local $U(1)_{\rm EM} \times U(1)_D$. We denote the gauge bosons by A_μ and C_μ, respectively.

(ii) There are four fermion fields:

$$e_L(-1,0), \qquad e_R(-1,0), \qquad d_L(0,-1), \qquad d_R(0,-1).$$

(iii) There is a single complex scalar:

$$\phi(q_{\rm EM}, q_D).$$

We assume no kinetic mixing and use a normalization such that the coupling constants of the two groups is the same, that is, $g_{\rm EM} = g_D = e$.

1. There are five specific charge assignments that allow Yukawa interactions, that is, couplings between ϕ and the fermions. What are these charge assignments?

From this point on, we do not consider any of the above options, that is, we consider only cases where all Yukawa interactions are forbidden.

2. Write the scalar potential. What is the condition for ϕ to acquire a VEV? From here on, assume that this condition is satisfied.
3. One way to make the model possibly consistent with Nature is to have partial SSB, such that the photon A_μ is massless but the dark photon C_μ is massive. Explain why this is the case when $q_{\rm EM} = 0$ and $q_D \neq 0$.
4. In the above case, that is with $q_{\rm EM} = 0$ and $q_D \neq 0$, write the mass of the dark photon in terms of the model parameters.
5. We now consider a case where both $q_{\rm EM} \neq 0$ and $q_D \neq 0$. In this case both $U(1)_{\rm EM}$ and $U(1)_D$ are broken. Show, however, that the breaking pattern is $[U(1)]^2 \to U(1)$. We denote the massless gauge boson A'_μ and the massive one C'_μ.